Journal of Statistical Physics, Vol. 83, Nos. 5/6, 1996

Analyticity and Integrability in the Chiral Potts Model

Barry M. McCoy[1] and William P. Orrick[1]

Received September 18, 1995; final November 1, 1995

We study the perturbation theory for the general nonintegrable chiral Potts model depending on two chiral angles and a strength parameter and show how the analyticity of the ground-state energy and correlation functions dramatically increases when the angles and the strength parameter satisfy the integrability condition. We further specialize to the superintegrable case and verify that a sum rule is obeyed.

KEY WORDS: Chiral Potts; integrability; analyticity; superintegrability; perturbation theory; Dedekind sum.

1. INTRODUCTION

The connection between integrability and analyticity has been recognized and exploited for many years in the theory of solvable models in statistical mechanics and in particular in the method of solution based on the inversion relation,[1,2] where assumptions on analyticity are a vital ingredient of the solution. Conversely, whenever an integrable model can be solved by other means the analyticity demanded by the inversion relation is always found to exist.

On the other hand, these analyticity statements have never been made particularly precise, and indeed there are recent computations in certain four-dimensional field theories,[3,4] where results may be obtained because functions also have dramatically expanded analyticity properties over what is generically expected, but no analog of the two-dimensional integrability conditions is known. Consequently it is of interest to sharpen our understanding of the connection between integrability and analyticity.

[1] Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11794-3840; e-mail: mccoy@max.physics.sunysb.edu, worrick@insti.physics.sunysb.edu.

0022-4715/96/0600-0839$09.50/0

In this paper we examine these questions for the N-state chiral Potts spin chain defined by

$$H_{\mathrm{CP}} = -\sum_{j=1}^{L}\sum_{n=1}^{N-1}\left(\frac{e^{i(2n-N)\phi/N}}{\sin(\pi n/N)}(Z_j Z_{j+1}^{\dagger})^n + k\frac{e^{i(2n-N)\bar{\phi}/N}}{\sin(\pi n/N)}X_j^n\right) \quad (1.1)$$

The matrices X_j, Z_j are defined by

$$X_j = I_N \otimes \cdots \underbrace{X}_{\text{site } j} \cdots \otimes I_N, \qquad Z_j = I_N \otimes \cdots \underbrace{Z}_{\text{site } j} \cdots \otimes I_N \quad (1.2)$$

where I_N is the $N \times N$ identity matrix, the elements of the $N \times N$ matrices X and Z are

$$X_{l,m} = \delta_{l,m+1} (\mathrm{mod}\ N), \qquad Z_{l,m} = \delta_{l,m}\omega^{l-1} \quad (1.3)$$

and $\omega = e^{2\pi i/N}$. When the angles ϕ and $\bar{\phi}$ and the strength parameter k satisfy the condition

$$\cos\phi = k\cos\bar{\phi} \quad (1.4)$$

this spin-chain Hamiltonian is derivable from the N-state chiral Potts model whose transfer matrix $T_{p,q}$ satisfies the integrability condition of commuting transfer matrices[5]

$$[T_{p,q}, T_{p,q'}] = 0 \quad (1.5)$$

where p and q specify points on the Riemann surface given by the intersection of two Fermat cylinders[6, 7]

$$a^N + kb^N = k'd^N, \qquad ka^N + b^N = k'c^N \quad (1.6)$$

with $k^2 + k'^2 = 1$. The Hamiltonian (1.1) is obtained from $T_{p,q}$ in the limit $q \to p$ as[8]

$$T_{p,q} = I + u(H_{\mathrm{CP}} + \mathrm{const}) + O(u^2) \quad (1.7)$$

where u measures the deviation of q from p and

$$e^{2i\phi/N} = \omega^{1/2}\frac{a_p c_p}{b_p d_p}, \qquad e^{2i\bar{\phi}/N} = \omega^{1/2}\frac{a_p d_p}{b_p c_p} \quad (1.8)$$

The free energy of the statistical mechanical system and the ground-state energy of the spin chain have been computed in the general integrable case[9] using the algebraic functional equation formalism of ref. 10 and in the special superintegrable case of $\phi = \bar{\phi} = \pi/2$ where there is a dramatic simplification.[8, 11–12] Our study here is to relate the general nonintegrable case to the integrable and superintegrable cases.

For the nonintegrable case the only analytic tool available is perturbation theory in the variable k. This expansion can be done both for $k \sim 0$ and $k \sim \infty$. In this paper we study only the first case and consider the expansion for the N-state spin chain of the ground-state energy and correlation functions. We observe that as the order increases, new singularities in the $e^{i\phi}$ plane continually arise and we expect that the number of singularities will become infinite as the order goes to infinity. The integrability condition (1.4), however, involves k and causes extensive cancellations among orders. We find that all but a finite number of singularities, which are already present in the lowest orders, vanish, resulting in a dramatic increase in analyticity. This increase in analyticity is intimately related to the integrability of the model.

In Section 2 we present the perturbation expansion of the ground-state energy for general chiral angles ϕ and $\bar{\phi}$ for $N = 3$–6 and then exhibit the increase in analyticity that occurs when the integrability condition (1.4) is imposed. In Section 3 the same procedure is applied to the correlation functions for $N = 3$. In addition, the perturbation expansion for the superintegrable correlation functions is given for arbitrary N. In Section 4 a sum rule for the nearest neighbor superintegrable correlation functions is derived and used to verify the expansion in Section 3. Also, a Fuchsian equation for the superintegrable ground-state energy is derived. In Section 5 we conclude with some questions for further study and some general remarks on the connection between integrability and analyticity. Some of the results in this paper were reported previously.[13]

2. PERTURBATION EXPANSION OF THE GROUND-STATE ENERGY

In this section we consider the expansion for the N-state chiral Potts spin chain with general chiral angles ϕ and $\bar{\phi}$ of the ground-state energy

$$e^{(N)} = \lim_{L \to \infty} E_{\mathrm{GS}}^{N}/L = \sum_{n=0}^{\infty} k^n e_n^{(N)} \tag{2.1}$$

We have implemented Rayleigh–Schrödinger perturbation theory[14] on the computer. A C program was used to enumerate all the spin-chain configurations contributing to a given order and to generate a recursively defined set of weights associated with these configurations. These were processed and combined using the Form symbolic manipulation program, and partial fractionation and final simplification were done using either

Maple or Mathematica. The following cases were studied: (a) $N=3$ to order 9, (b) $N=4$ to order 8, (c) $N=5$ to order 5, and (d) $N=6$ to order 6. Our results for these four cases are given in Tables I–IV, were we use the following notation:

$$C_3=\cos(\phi/3), \qquad C_4=\sqrt{2}\cos(\phi/2)$$
$$C_5=\frac{\cos(\phi/5)}{\sin(\pi/5)}=\sqrt{2}\,(1+5^{-1/2})^{1/2}\cos(\phi/5), \qquad C_6=\sqrt{3}\cos(\phi/3) \tag{2.2}$$

In the case of $N=3$ there have been previous, independent perturbation studies both for small[15,16] and large[16–18] k and Table I agrees with the small-k results of ref. 15, which were obtained to order 7. Many of the details of the perturbation expansion are discussed in these papers.

These tables demonstrate an important feature of the generic perturbation theory; namely that as the order of perturbation theory increases, so do the number of places in the $e^{i\phi}$ plane where the perturbation expansion develops singularities. Thus, for example, we see from Table I that $e_2^{(3)}$ and $e_3^{(3)}$ are singular only when $C_3=0$, but that $e_4^{(3)}$, $e_5^{(3)}$, $e_6^{(3)}$, and $e_7^{(3)}$ have additional singularities at $C_3=\pm 1/2$ and $e_8^{(3)}$ and $e_9^{(3)}$ have further singularities at $28C_3^2-3=0$. A similar increase in number of singularities is seen in Table II. Indeed this property of the number of singularities increasing with the order has been explicitly seen in a variety of previous perturbation computations, including the Ising model in a magnetic field[1] and the zero-field Ising magnetic susceptibility.[19] In the latter case is it well understood that the number of singularities is connected to the number of quasiparticles that contribute to the given order of perturbation theory[20,21] and that as the order of perturbation goes to infinity, an infinite number of singularities will appear. It is this ever-increasing number of singularities in the perturbation theory which in the case of the Ising model in a magnetic field restricts the analyticity to such an extent[1] that a closed-form solution has never been found.

However, there is a dramatic change in this pattern of singularities when the parameters ϕ, $\bar{\phi}$, and k are related by the integrability condition (1.4). This is possible because this integrability condition (1.4) involves the perturbation parameter k and thus allows different orders of perturbation theory to be combined.

To appreciate the cancellations between the orders of perturbation theory that result, we consider $N=3$ in detail. It is easy to see that when (1.4) holds, if we use

$$\frac{\cos\phi}{\cos(\phi/3)}=e^{2i\phi/3}-1+e^{-2i\phi/3} \tag{2.3}$$

Table I. The First Nine Orders of Perturbation in k for $N=3$ for the Ground-State Energy in the Generic Case[a]

$$e_0^{(3)} = -\frac{4C_3}{\sqrt{3}}, \quad e(3)_1 = 0, \quad e_2^{(3)} = \frac{-2}{3\sqrt{3}\,C_3}$$

$$e_3^{(3)} = \frac{-\cos\bar{\phi}}{9\sqrt{3}\,C_3^2}$$

$$e_4^{(3)} = \sqrt{3}\left(\frac{1}{162C_3^3} + \frac{4}{81C_3} - \frac{4}{81(-1+2C_3)} - \frac{4}{81(1+2C_3)}\right)$$

$$e_5^{(3)} = -\left(\sqrt{3}\left(\frac{-\cos\bar{\phi}}{108C_3^4} - \frac{4\cos\bar{\phi}}{81C_3^2} + \frac{8\cos\bar{\phi}}{81(-1+2C_3)} - \frac{8\cos\bar{\phi}}{81(1+2C_3)}\right)\right)$$

$$\begin{aligned}
e_6^{(3)} = \sqrt{3}\Bigg(&\frac{-2}{729(-1+2C_3)^3} - \frac{1}{729(-1+2C_3)^2} + \frac{2(1-54\cos^2\bar{\phi})}{2187(-1+2C_3)} \\
&- \frac{2}{729(1+2C_3)^3} + \frac{1}{729(1+2C_3)^2} + \frac{2(1-54\cos^2\bar{\phi})}{2187(1+2C_3)} \\
&+ \frac{11+24\cos^2\bar{\phi}}{11664C_3^5} + \frac{5+54\cos^2\bar{\phi}}{4374C_3^3} + \frac{2(-1+54\cos^2\bar{\phi})}{2187C_3}\Bigg)
\end{aligned}$$

$$\begin{aligned}
e_7^{(3)} = -\sqrt{3}\Bigg(&\frac{\cos\bar{\phi}}{972C_3^6} + \frac{263\cos\bar{\phi}}{26244C_3^4} + \frac{1022\cos\bar{\phi}}{19683C_3^2} \\
&+ \frac{\cos\bar{\phi}}{2187(-1+2C_3)^4} + \frac{64\cos\bar{\phi}}{6561(-1+2C_3)^3} - \frac{119\cos\bar{\phi}}{39366(-1+2C_3)^2} - \frac{49\cos\bar{\phi}}{486(-1+2C_3)} \\
&+ \frac{\cos\bar{\phi}}{2187(1+2C_3)^4} - \frac{64\cos\bar{\phi}}{6561(1+2C_3)^3} - \frac{119\cos\bar{\phi}}{39366(1+2C_3)^2} + \frac{49\cos\bar{\phi}}{486(1+2C_3)}\Bigg)
\end{aligned}$$

$$\begin{aligned}
e_8^{(3)} = \sqrt{3}\Bigg(&\frac{-1}{2187(-1+2C_3)^5} + \frac{-1-2\cos^2\bar{\phi}}{2187(-1+2C_3)^4} + \frac{55-888\cos^2\bar{\phi}}{78732(-1+2C_3)^3} \\
&+ \frac{-451+1384\cos^2\bar{\phi}}{157464(-1+2C_3)^2} + \frac{2759+15264\cos^2\bar{\phi}}{78732(-1+2C_3)} \\
&- \frac{1}{2187(1+2C_3)^5} + \frac{1+2\cos^2\bar{\phi}}{2187(1+2C_3)^4} + \frac{55-888\cos^2\bar{\phi}}{78732(1+2C_3)^3} + \frac{451-1384\cos^2\bar{\phi}}{157464(1+2C_3)^2} \\
&+ \frac{2759+15264\cos^2\bar{\phi}}{78732(1+2C_3)} - \frac{1715C_3}{2187(-3+28C_3^2)} \\
&+ \frac{-381-1936\cos^2\bar{\phi}}{1679616C_3^7} + \frac{-801-6208\cos^2\bar{\phi}}{629856C_3^5} \\
&+ \frac{-709-7880\cos^2\bar{\phi}}{157464C_3^3} + \frac{-277-7632\cos^2\bar{\phi}}{39366C_3}\Bigg)
\end{aligned}$$

[a] We use the variable $C_3 = \cos(\phi/3)$.

Table I. *(Continued)*

$$e_9^{(3)} = -\sqrt{3}\left(\frac{5\cos\bar\phi}{26244(-1+2C_3)^6}+\frac{17\cos\bar\phi}{8748(-1+2C_3)^5}+\frac{61\cos\bar\phi+432\cos^3\bar\phi}{944784(-1+2C_3)^4}\right.$$
$$+\frac{-11257\cos\bar\phi+6048\cos^3\bar\phi}{1417176(-1+2C_3)^3}+\frac{105767\cos\bar\phi-24912\cos^3\bar\phi}{5668704(-1+2C_3)^2}$$
$$+\frac{-34607\cos\bar\phi-59184\cos^3\bar\phi}{629856(-1+2C_3)}+\frac{5\cos\bar\phi}{26244(1+2C_3)^6}-\frac{17\cos\bar\phi}{8748(1+2C_3)^5}$$
$$+\frac{61\cos\bar\phi+432\cos^3\bar\phi}{944784(1+2C_3)^4}+\frac{11257\cos\bar\phi-6048\cos^3\bar\phi}{1417176(1+2C_3)^3}$$
$$+\frac{105767\cos\bar\phi-24912\cos^3\bar\phi}{5668704(1+2C_3)^2}+\frac{34607\cos\bar\phi+59184\cos^3\bar\phi}{629856(1+2C_3)}$$
$$+\frac{24010\cos\bar\phi}{19683(-3+28C_3^2)}$$
$$+\frac{133\cos\bar\phi+192\cos^3\bar\phi}{839808C_3^8}+\frac{13(11\cos\bar\phi+52\cos^3\bar\phi)}{314928C_3^6}$$
$$\left.+\frac{-575\cos\bar\phi+3696\cos^3\bar\phi}{314928C_3^4}+\frac{-9007\cos\bar\phi+17424\cos^3\bar\phi}{354294C_3^2}\right)$$

Table II. The First Eight Orders of Perturbation in k for $N=4$ for the Ground-State Energy in the Generic Case[a]

$$e_0^{(4)} = -1-2C_4,\quad e_1^{(4)}=0,\quad e_2^{(4)}=\frac{-1}{8C_4}-\frac{1}{1+C_4}$$

$$e_3^{(4)} = -\frac{\cos\bar\phi}{4C_4}-\frac{\cos\bar\phi}{4(1+C_4)^2}+\frac{\cos\bar\phi}{4(1+C_4)}$$

$$e_4^{(4)} = \frac{-1}{512C_4^3}+\frac{1}{16C_4^2}+\frac{5-2\cos^2\bar\phi}{16C_4}+\frac{1}{12(1+C_4)^3}+\frac{8+9\cos^2\bar\phi}{72(1+C_4)^2}$$
$$+\frac{20+27\cos^2\bar\phi}{216(1+C_4)}-\frac{875}{432(-1+5C_4)}$$

$$e_5^{(4)} = \frac{3\cos\bar\phi}{128C_4^3}+\frac{19\cos\bar\phi}{64C_4^2}+\frac{149\cos\bar\phi}{128C_4}+\frac{61\cos\bar\phi}{576(1+C_4)^4}+\frac{55\cos\bar\phi}{864(1+C_4)^3}$$
$$+\frac{\cos\bar\phi}{1152(1+C_4)^2}-\frac{277\cos\bar\phi}{31104(1+C_4)}-\frac{1225\cos\bar\phi}{648(-1+5C_4)^2}-\frac{89825\cos\bar\phi}{15552(-1+5C_4)}$$

[a] We use the variable $C_4=\sqrt{2}\cos(\phi/2)$.

Table II. *(Continued)*

$$e_6^{(4)}=\frac{2025}{8192(-1-3C_4)}-\frac{1}{8192C_4^5}+\frac{3}{512C_4^4}+\frac{-7+52\cos^2\bar{\phi}}{1024C_4^3}+\frac{3(-23+48\cos^2\bar{\phi})}{512C_4^2}$$
$$+\frac{-303+812\cos^2\bar{\phi}}{1024C_4}+\frac{155-44\cos^2\bar{\phi}}{3456(1+C_4)^5}+\frac{-33-394\cos^2\bar{\phi}}{3456(1+C_4)^4}+\frac{-253-3911\cos^2\bar{\phi}}{20736(1+C_4)^3}$$
$$+\frac{-15832-72167\cos^2\bar{\phi}}{373248(1+C_4)^2}+\frac{75203-236166\cos^2\bar{\phi}}{1492992(1+C_4)}-\frac{8575(5+4\cos^2\bar{\phi})}{15552(-1+5C_4)^3}$$
$$+\frac{175(8345-12704\cos^2\bar{\phi})}{746496(-1+5C_4)^2}+\frac{25(727595-758184\cos^2\bar{\phi})}{5971968(-1+5C_4)}$$
$$+\frac{9(1+C_4)}{16(1-2C_4-2C_4^2)}$$

$$e_7^{(4)}=\frac{5\cos\bar{\phi}}{4096C_4^5}-\frac{31\cos\bar{\phi}}{1024C_4^4}-\frac{1301\cos\bar{\phi}-96\cos^3\bar{\phi}}{4096C_4^3}-\frac{309\cos\bar{\phi}+20\cos^3\bar{\phi}}{256C_4^2}$$
$$-\frac{3855\cos\bar{\phi}+2944\cos^3\bar{\phi}}{4096C_4}-\frac{101\cos\bar{\phi}+960\cos^3\bar{\phi}}{41472(1+C_4)^6}$$
$$-\frac{13385\cos\bar{\phi}+5904\cos^3\bar{\phi}}{124416(1+C_4)^5}$$
$$-\frac{124655\cos\bar{\phi}+83952\cos^3\bar{\phi}}{1492992(1+C_4)^4}-\frac{96257\cos\bar{\phi}+14169\cos^3\bar{\phi}}{559872(1+C_4)^3}$$
$$-\frac{4392121\cos\bar{\phi}-488304\cos^3\bar{\phi}}{26873856(1+C_4)^2}$$
$$-\frac{11116697\cos\bar{\phi}-1116063\cos^3\bar{\phi}}{20155392(1+C_4)}+\frac{1539\cos\bar{\phi}}{16384(1+3C_4)^2}-\frac{25839\cos\bar{\phi}}{32768(1+3C_4)}$$
$$-\frac{1860775\cos\bar{\phi}}{186624(-1+5C_4)^4}-\frac{1225(4103\cos\bar{\phi}+2800\cos^3\bar{\phi})}{746496(-1+5C_4)^3}$$
$$-\frac{25(-47222099\cos\bar{\phi}+621600\cos^3\bar{\phi})}{35831808(-1+5C_4)^2}$$
$$+\frac{25(432042539\cos\bar{\phi}+85572000\cos^3\bar{\phi})}{644972544(-1+5C_4)}-\frac{51(2\cos\bar{\phi}+\cos\bar{\phi}\,C_4)}{16(-1+2C_4+2C_4^2)}$$

$$e_8^{(4)}=\frac{81(199+32\cos^2\bar{\phi})}{131072(-1-3C_4)^3}+\frac{81(-803+7760\cos^2\bar{\phi})}{1048576(-1-3C_4)^2}$$
$$+\frac{81(-20863+102076\cos^2\bar{\phi})}{13631488(-1-3C_4)}$$
$$-\frac{25}{2097152C_4^7}+\frac{9}{16384C_4^6}+\frac{-151-158\cos^2\bar{\phi}}{16384C_4^5}$$

Table II. *(Continued)*

$$+\frac{-519-3008\cos^2\bar{\phi}}{16384C_4^4}+\frac{359-15946\cos^2\bar{\phi}-32\cos^4\bar{\phi}}{16384C_4^3}$$

$$+\frac{7923-43936\cos^2\bar{\phi}-2048\cos^4\bar{\phi}}{16384C_4^2}+\frac{76123+14074\cos^2\bar{\phi}-10816\cos^4\bar{\phi}}{16384C_4}$$

$$+\frac{-53297+31488\cos^2\bar{\phi}-4608\cos^4\bar{\phi}}{1990656(1+C_4)^7}$$

$$+\frac{-555061-193632\cos^2\bar{\phi}+138240\cos^4\bar{\phi}}{23887872(1+C_4)^6}$$

$$+\frac{-1933609+12161664\cos^2\bar{\phi}+2257920\cos^4\bar{\phi}}{95551488(1+C_4)^5}$$

$$+\frac{85101083-62007936\cos^2\bar{\phi}+179776512\cos^4\bar{\phi}}{3439853568(1+C_4)^4}$$

$$+\frac{-2291995019+25564353792\cos^2\bar{\phi}+3332192256\cos^4\bar{\phi}}{41278242816(1+C_4)^3}$$

$$+\frac{9085861571-47135217152\cos^2\bar{\phi}+5619560448\cos^4\bar{\phi}}{55037657088(1+C_4)^2}$$

$$+\frac{-917428110137+7122467770368\cos^2\bar{\phi}+222666719232\cos^4\bar{\phi}}{1981355655168(1+C_4)}$$

$$-\frac{420175(2+3\cos^2\bar{\phi})}{69984(-1+5C_4)^5}+\frac{8575(-3751+21936\cos^2\bar{\phi})}{26873856(-1+5C_4)^4}$$

$$+\frac{175(161445421+287088144\cos^2\bar{\phi}-35280000\cos^4\bar{\phi})}{2579890176(-1+5C_4)^3}$$

$$+\frac{175(-835186517+7197054312\cos^2\bar{\phi}+37920000\cos^4\bar{\phi})}{10319560704(-1+5C_4)^2}$$

$$+\frac{125(1145946953233+5452562686272\cos^2\bar{\phi}+282187776000\cos^4\bar{\phi})}{12878811758592(-1+5C_4)}$$

$$-\frac{29176875}{33554432(1+5C_4)}+\frac{9(4+3C_4)}{128(1-2C_4-2C_4^2)^3}-\frac{3(81+46C_4)}{512(1-2C_4-2C_4^2)^2}$$

$$+\frac{-19777+197496\cos^2\bar{\phi}-22351C_4+197496\cos^2\bar{\phi}\,C_4}{6656(1-2C_4-2C_4^2)}$$

$$-\frac{27(48104891+84022143C_4)}{16777216(-1+8C_4+17C_4^2)}$$

Table III. The First Five Orders of Perturbation in k for $N=5$ for the Ground-State Energy in the Generic Case[a]

$$e_0^{(5)} = -(5-\sqrt{5})\,C_5^3 + (7-\sqrt{5})\,C_5, \quad e_1^{(5)} = 0$$

$$e_2^{(5)} = (5+\sqrt{5})\left(\frac{-(3+\sqrt{5})}{50(-1+C_5)} + \frac{9+\sqrt{5}}{50C_5} - \frac{3+\sqrt{5}}{50(1+C_5)} + \frac{(-3+\sqrt{5})\,C_5}{50(-2+C_5^2)}\right)$$

$$e_3^{(5)} = \frac{(5+\sqrt{5})^{1/2}}{-5+3\sqrt{5}}\left(\frac{-((5+3\sqrt{5})\cos\bar{\phi})}{125\sqrt{2}(-1+C_5)^2} + \frac{(35+13\sqrt{5})\cos\bar{\phi}}{125\sqrt{2}(-1+C_5)} - \frac{\sqrt{2}(25+6\sqrt{5})\cos\bar{\phi}}{125C_5^2}\right.$$
$$- \frac{(5+3\sqrt{5})\cos\bar{\phi}}{125\sqrt{2}(1+C_5)^2} - \frac{(35+13\sqrt{5})\cos\bar{\phi}}{125\sqrt{2}(1+C_5)} + \frac{2\sqrt{2}(-5+2\sqrt{5})\cos\bar{\phi}}{125(-2+C_5^2)^2}$$
$$\left.- \frac{\sqrt{2}(5+4\sqrt{5})\cos\bar{\phi}}{125(-2+C_5^2)}\right)$$

$$e_4^{(5)} = \cos^2\bar{\phi}\left(\frac{110+50\sqrt{5}}{3125(-1+C_5)^2} - \frac{4(15+7\sqrt{5})}{625(-1+C_5)}\right.$$
$$+ \frac{260+120\sqrt{5}}{3125C_5^3} + \frac{630+290\sqrt{5}}{3125C_5} - \frac{110+50\sqrt{5}}{3125(1+C_5)^2} - \frac{4(15+7\sqrt{5})}{625(1+C_5)}$$
$$\left.+ \frac{(-40-20\sqrt{5})\,C_5}{3125(-2+C_5^2)^2} - \frac{(30+10\sqrt{5})\,C_5}{3125(-2+C_5^2)}\right) + \cdots$$

$$e_5^{(5)} = \left(\frac{-52(10-2\sqrt{5})^{1/2}}{3125} - \frac{116(10-2\sqrt{5})^{1/2}}{3125\sqrt{5}}\right)\cos^3\bar{\phi}\left(\frac{1}{4(-1+C_5)^2} - \frac{1}{4(-1+C_5)}\right.$$
$$\left.+ \frac{1}{4C_5^4} + \frac{3}{4C_5^2} + \frac{1}{4(1+C_5)^2} + \frac{1}{4(1+C_5)} + \frac{1}{4(-2+C_5^2)^2} - \frac{3}{4(-2+C_5^2)}\right) + \cdots$$

[a] For the fourth and fifth orders only the terms proportional to $\cos^2\bar{\phi}$ and $\cos^3\bar{\phi}$, respectively, are given. We use the variable $C_5 = \cos(\phi/5)/\sin(\pi/5)$.

Table IV. The First Six Orders of Perturbation in k for $N=6$ for the Ground-State Energy in the Generic Case[a]

$$e_0^{(6)} = 3 - \frac{4}{3}C_6 - \frac{8}{3}C_6^2, \quad e_1^{(6)} = 0$$

$$e_2^{(6)} = \frac{-2}{15(-1+C_6)} - \frac{2}{C_6} + \frac{64}{15(3+2C_6)} - \frac{3}{4(-9+8C_6^2)}$$

$$e_3^{(6)} = \frac{-\cos\bar{\phi}}{75\sqrt{3}(-1+C_6)^2} + \frac{94\cos\bar{\phi}}{375\sqrt{3}(-1+C_6)} - \frac{\cos\bar{\phi}}{\sqrt{3}C_6^2} + \frac{8\cos\bar{\phi}}{3\sqrt{3}C_6} - \frac{484\cos\bar{\phi}}{75\sqrt{3}(3+2C_6)^2}$$
$$- \frac{132\sqrt{3}\cos\bar{\phi}}{125(3+2C_6)} - \frac{32\cos\bar{\phi}\,C_6}{3\sqrt{3}(-9+8C_6^2)}$$

[a] For the sixth order only the term proportional to $\cos^4\bar{\phi}$ is given. We use the variable $C_5 = \sqrt{3}\cos(\phi/3)$.

Table IV. *(Continued)*

$$e_4^{(6)} = \frac{1}{2250(-1+C_6)^3} + \frac{667+360\cos^2\bar\phi}{15000(-1+C_6)^2} + \frac{216617+527040\cos^2\bar\phi}{450000(-1+C_6)}$$
$$+\frac{11}{30C_6^3} + \frac{2(-39+50\cos^2\bar\phi)}{225C_6^2} - \frac{-3691+5000\cos^2\bar\phi}{6750C_6}$$
$$+\frac{2(-2381+3060\cos^2\bar\phi)}{3375(3+2C_6)^3} + \frac{-1620919+579960\cos^2\bar\phi}{911250(3+2C_6)^2}$$
$$-\frac{708766379+423636480\cos^2\bar\phi}{492075000(3+2C_6)}$$
$$-\frac{1294139}{243000(-15+14C_6)} - \frac{2(3+C_6)}{99(-4+C_6+2C_6^2)} - \frac{4(9+8C_6)}{297(-3+4C_6^2)}$$
$$-\frac{27}{64(-9+8C_6^2)^3} + \frac{4(-9+16C_6)}{9(-9+8C_6^2)^2} - \frac{44(-11+36\cos^2\bar\phi+8C_6)}{81(-9+8C_6^2)}$$
$$+\frac{9878103-19929087C_6+7810406C_6^2}{314928(27-36C_6-33C_6^2+38C_6^3)}$$

$$e_5^{(6)} = \frac{\cos\bar\phi}{1000\sqrt{3}(-1+C_6)^3} + \frac{193\cos\bar\phi}{10000\sqrt{3}(-1+C_6)^2}$$
$$+\frac{\cos\bar\phi(-242459+194400\cos^2\bar\phi)}{600000\sqrt{3}(-1+C_6)} + \frac{5427\sqrt{3}\cos\bar\phi}{160C_6^3} - \frac{126279\sqrt{3}\cos\bar\phi}{800C_6^2}$$
$$-\frac{3\sqrt{3}\cos\bar\phi(-185083+11250\cos^2\bar\phi)}{2000C_6} - \frac{81\sqrt{3}\cos\bar\phi(-65789+5400\cos^2\bar\phi)}{1000(3+2C_6)^3}$$
$$+\frac{9\sqrt{3}\cos\bar\phi(-842051+361575\cos^2\bar\phi)}{2500(3+2C_6)^2}$$
$$+\frac{\cos\bar\phi(-2735083651+191181600\cos^2\bar\phi)}{800000\sqrt{3}(3+2C_6)} + \frac{8349\sqrt{3}\cos\bar\phi}{32000(-15+14C_6)}$$
$$-\frac{469\cos\bar\phi(-302+255C_6)}{186\sqrt{3}(-4+C_6+2C_6^2)} + \frac{\sqrt{3}\cos\bar\phi(-229+264C_6)}{8(-3+4C_6^2)}$$
$$+\frac{3645\sqrt{3}\cos\bar\phi(-123+116C_6)}{1024(-9+8C_6^2)^3} - \frac{27\sqrt{3}\cos\bar\phi(-83589+79246C_6)}{512(-9+8C_6^2)^2}$$
$$-\frac{9\sqrt{3}\cos\bar\phi(8025965-2928384\cos^2\bar\phi-8380680C_6+2761728\cos^2\bar\phi\, C_6)}{31744(-9+8C_6^2)}$$
$$+\frac{\cos\bar\phi(22455-53739C_6+29648C_6^2)}{4\sqrt{3}(27-36C_6-33C_6^2+38C_6^3)}$$

$$e_6^{(6)} = \cos^4\bar\phi\left(\frac{27}{625(-1+C_6)^2} + \frac{1674}{3125(-1+C_6)} + \frac{1}{27C_6^2} - \frac{2}{81C_6} - \frac{16}{75(3+2C_6)^4}\right.$$
$$\left.-\frac{64}{125(3+2C_6)^3} - \frac{13312}{16875(3+2C_6)^2} - \frac{258688}{253125(3+2C_6)} - \frac{256}{27(-9+8C_6^2)}\right) + \cdots$$

Table V. The Results of Specializing the Generic Results of the Perturbation Theory for the Ground-State Energy for $N=3$, 4, 5, and 6 to the Integrable Manifold (1.4) Where $\cos\phi = k\cos\bar{\phi}$

$$e^{(3)} = \frac{-4C_3}{\sqrt{3}} - k^2\frac{3+4C_3^2}{9\sqrt{3}\,C_3} - k^4\frac{3+16C_3^4}{324\sqrt{3}\,C_3^3} - k^6\frac{27+108C_3^2-72C_3^4+800C_3^6}{52488\sqrt{3}\,C_3^5} + O(k^8)$$

$$e^{(4)} = -1-2C_4-k^2\left(\frac{1}{4}+\frac{C_4}{8}+\frac{1}{2(1+C_4)}\right) - k^4\left(\frac{1}{64}+\frac{9C_4}{512}+\frac{1}{32(1+C_4)^3}+\frac{3}{64(1+C_4)^2}\right)+O(k^6)$$

$$e^{(5)} = -(5-\sqrt{5})\,C_5^3+(7-\sqrt{5})\,C_5 + k^2\left(\frac{1}{5(-5+2\sqrt{5})(-1+C_5)}+\frac{2(-3+\sqrt{5})}{5(-5+2\sqrt{5})\,C_5}+\frac{1}{5(-5+2\sqrt{5})(1+C_5)} - \frac{C_5(18-8\sqrt{5}-7C_5^2+3\sqrt{5}\,C_5^2)}{5(-5+2\sqrt{5})}\right)+O(k^4)$$

$$e^{(6)} = 3-\frac{4}{3}C_6-\frac{8}{3}C_6^2-k^2\left(\frac{2}{27}C_6^2+\frac{4}{27}C_6+\frac{5}{36}+\frac{1}{C_6}-\frac{4}{3(3+2C_6)}\right)+O(k^4)$$

then $e_2^{(3)}$ and $e_3^{(3)}$ are combined as

$$k^2e_2^{(3)}+k^3e_3^{(3)} = -k^2\frac{3+4C_3^2}{9\sqrt{3}\,C_3} \tag{2.4}$$

which is the k^2 term on the integrable manifold as given in Table V.

A more important cancellation, however, takes place between the terms $e_4^{(3)}$, $e_5^{(3)}$, and $e_6^{(3)}$ to give the k^4 term in Table V. Here the terms in $\cos\bar{\phi}$ in the k^5 terms and the terms in $\cos^2\bar{\phi}$ in the k^6 term which have the denominators $\pm 1+2C_3$ combine with the terms in the k^4 term with the same denominators to cancel the denominators out completely. In addition, the terms with C_3^{-4} in $e_5^{(3)}$ and with C_3^{-5} in $e_6^{(3)}$ are reduced to C_3^{-3} by use of (2.3) and thus the complete k^4 term in Table V is obtained from three terms in Table I. Similarly, the terms in $e_6^{(3)}$ which have no factor of $\cos\bar{\phi}$ combine with the terms in $e_7^{(3)}$ with $\cos\bar{\phi}$, the terms in e_8 with $\cos^2\bar{\phi}$, and the terms in $e_9^{(3)}$ with $\cos^3\bar{\phi}$ to cancel all denominators containing powers of $\pm 1+2C_3$. Thus we obtain the term of order k^6 in Table V.

For $N=4$ we find from Table II that the terms of orders 2, 3, and 4 are needed to get the k^2 term on the integrable manifold and that to get the k^4 term in the integrable case the terms in the generic case to order 8 are needed. In general, to get the order-k^{2q} term in the integrable case the terms to order Nq of the generic case are needed. To show this we first note that $e^{(N)}$ is an even function of $\bar{\phi}$. Second, the order-r term in the generic case is a sum of terms of the form

$$C_{n_1,\ldots,n_r;j_1,\ldots,j_r}(\phi)\,k^r \exp\left(\frac{i\bar{\phi}}{N}\left(2\sum_{l=1}^{r} n_l - Nr\right)\right)\langle_0 0|\prod_{l=1}^{r} X_{j_l}^{n_l}|0\rangle_0 \quad (2.5)$$

where $|0\rangle_0$ is the ground state when $k=0$. The last term in (2.5) clearly vanishes if $\sum_{l=1}^{r} n_l$ is not an integer multiple of N. Thus the k^r term can be written

$$e^{(N)} = \sum_{r=0}^{\infty} k^r \sum_{p=0}^{p_{\max}}{}' C_p(\phi)\cos(p\bar{\phi}) = \sum_{r=0}^{\infty} k^r \sum_{p=0}^{p_{\max}}{}' C'_p(\phi)\cos^p\bar{\phi} \quad (2.6)$$

where $\sum'$ indicates summation over even values if r is even and odd values if r is odd. Since the maximum allowed value for the n_l in (2.5) is $N-1$, the maximum allowed value of p for a given r is $p_{\max}(r) = 2\lfloor r(N-1)/N\rfloor - r$ where $[X]$ is the greatest integer less than or equal to X. When the integrability condition (1.4) is applied we obtain

$$e^{(N)} = \sum_{r=0}^{\infty} \sum_{p=0}^{p_{\max}}{}' k^{r-p} C'_p(\phi)\cos^p\phi \quad (2.7)$$

which contains only even powers of k. The only terms in the generic case contributing to the k^{2q} term in the integrable case are those terms for which $r - p_{\max}(r) \leqslant 2q$, which is equivalent to the condition $-\lfloor -r/N\rfloor \leqslant q$ or $r \leqslant qN$, as desired.

The results of the specialization to the integrable manifold are given in Table V for the cases $N=3$–6. Thus we see in the cases considered that all singularities have cancelled out except certain of the original singularities in $e_2^{(N)}$. Consequently the analyticity of the ground-state energy has increased to the maximum extent possible. This remarkable extension in analyticity is a consequence of the defining commutation relation of integrability (1.5) and the relation (1.8) which the chiral angles have with the Riemann surface (1.6).

The integrable results are now easily specialized to the superintegrable case $\phi = \pi/2$ and we obtain

$$\begin{aligned} e^{(3)} &= -2 - \frac{4}{3^2}k^2 - \frac{8}{3^5}k^4 - \frac{20}{3^7}k^6 - O(k^8) \\ e^{(4)} &= -3 - \frac{5}{2^3}k^2 - \frac{25}{2^9}k^4 - O(k^6) \\ e^{(5)} &= -4 - \frac{4}{5}k^2 - O(k^4) \\ e^{(6)} &= -5 - \frac{35}{36}k^2 - O(k^4) \end{aligned} \quad (2.8)$$

which agrees with the previous computation of ref. 11. We note that in principle we should compare the integrable results of Table V with the exact result of ref. 9, but no systematic expansion of the exact result seems to be in the literature.

3. PERTURBATION EXPANSION OF THE CORRELATION FUNCTIONS

The cancellation of singularities and resulting expansion of analyticity seen in the ground state is, of course, expected to occur in all the correlation functions as well. However, the presentation of the results of the perturbation theory in the generic case is more cumbersome and consequently we will restrict ourselves here to illustrating the phenomenon for the case $N=3$.

The perturbation expansion of the correlation function $\langle Z_0 Z_r^\dagger \rangle$ for generic values of ϕ and $\bar{\phi}$ was studied in refs. 15 and 16 to fourth order. In order to exhibit the desired cancellations of the poles at $C_3 = \pm 1/2$, which occurred in the ground-state energy by combining the fourth, fifth, and sixth orders, we must extend the computation of the correlation to sixth order also. This we have done and the results are given in Table VI. The method of perturbation is discussed in some detail in refs. 15 and 16. The only additional remark we wish to make is that in contrast to more standard perturbation expansions, we have been unable to find an efficient way to perform the subtraction of the disconnected diagrams to obtain a general formalism that deals only with order-one objects in the $L \to \infty$ limit and are forced to perform an explicit subtraction of L-dependent terms. This seems to be a consequence of doing perturbation theory on a lattice with a variable that satisfies a cyclicity condition $X_j^N = 1$ and, because of the connection of the cyclicity condition with the fractional statistics of the model, it deserves further study.

To study the specialization of the generic result of Table VI to the integrable case (1.4) we consider first the long-range-order term $M^2 = \lim_{r \to \infty} \langle Z_0 Z_r^\dagger \rangle$, which in the perturbation expansion is the coefficient of $1 - \delta_{n,0}$. From the second- and third-order terms it is easy to see that the cancellation of poles is more complete than in the ground-state energy and that to second order on the integrable manifold we have the simplification to $-2k^2/9$, where the dependence on the chiral angle ϕ has completely vanished and thus the order parameter on the integrable manifold is identical with the order parameter on the superintegrable special case. This complete cancellation occurs in all orders and is a well-known consequence[22, 23] of the integrability commutation relation (1.5). These perturbation computations for the long-range order have been carried to high

Table VI. The First Six Orders of Perturbation Theory in k for $N=3$ for the Correlation Function $\langle Z_0 Z_r^\dagger \rangle = \sum_{n=0} k^n c_n^z$ in the Generic Case[a]

$$c_0^z = 1, \quad c_1^z = 0, \quad c_2^z = \frac{1}{6C_3^2}(\delta_{r,0}-1), \quad c_3^z = \frac{\cos\bar{\phi}}{18C_3^3}(\delta_{r,0}-1)$$

$$c_4^z = \left(\frac{1}{432C_3^4} + \frac{2}{81C_3^2} - \frac{2}{27(1-2C_3)^2} - \frac{2}{81(1-2C_3)} - \frac{2}{27(2C_3+1)^2} - \frac{2}{81(2C_3+1)}\right)(1-\delta_{r,0})$$
$$+\left(\frac{5}{432C_3^4} + \frac{1}{81C_3^2} + \frac{2iS_3}{27C_3} + \frac{1-2iS_3}{54(2C_3+1)^2} + \frac{7-12iS_3}{162(2C_3+1)}\right.$$
$$\left.+\frac{1+2iS_3}{54(2C_3-1)^2} + \frac{7+12iS_3}{162(2C_3-1)}\right)\delta_{r,1}$$

$$c_5^z = \cos\bar{\phi}\left(\frac{1}{648C_3^5} - \frac{1}{81C_3^3} + \frac{2}{81C_3} + \frac{2iS_3}{27C_3^2} + \frac{1+2iS_3}{27(2C_3-1)^2}\right.$$
$$\left.-\frac{2+18iS_3}{81(2C_3-1)} + \frac{-1+2iS_3}{27(2C_3+1)^2} + \frac{-2+18iS_3}{2C_3+1}\right)\delta_{r,1} + \cdots$$

$$c_6^z = \cos^2\bar{\phi}\left(\frac{-1}{1296C_3^6} - \frac{1}{108C_3^4} - \frac{1}{54C_3^2} + \frac{iS_3}{54C_3^3} + \frac{4iS_3}{27C_3}\right.$$
$$\left.+\frac{1-2iS_3}{54(2C_3+1)^2} - \frac{1+8iS_3}{54(2C_3+1)} + \frac{1+2iS_3}{54(2C_3-1)^2} + \frac{1-8iS_3}{54(2C_3-1)}\right)\delta_{r,1} + \cdots$$

[a] We use the notation $C_3 = \cos(\phi/3)$ and $S_3 = \sin(\phi/3)$. For c_5^z only the coefficient of $\delta_{r,1}$ and for c_6^z only the coefficient of $\cos^2\bar{\phi}\,\delta_{r,1}$ is given.

order in k for $N=3$ in refs. 23 and 24 and have been extended to arbitrary N in ref. 11. These expansions have led to the remarkably simple conjecture for the general case that

$$M_n^2 = \lim_{r\to\infty} \langle (Z_0 Z_r^\dagger)^n \rangle = (1-k^2)^{n(N-n)/N^2} \tag{3.1}$$

To see the first dependence on the variable ϕ in the correlation function on the integrable manifold we must go to fourth order, which is obtained from the fourth, fifth, and sixth orders of the generic result of Table VI. (Note that to obtain the k^{2q} term in the integrable case we need terms up to k^{Nq} in the generic case, just as we did for the ground-state energy.) In fourth order the only deviation of $\langle Z_0 Z_r^\dagger \rangle$ from M_1^2 occurs for $r=0, 1$ and since the case $r=0$ is trivial, we consider only $r=1$ and find that the poles at $C_3 = \pm 1/2$ cancel. Thus the result is obtained on the integrable manifold (1.4) of

$$\langle Z_0 Z_1^\dagger \rangle - M_1^2 = k^4\left(\frac{7}{162} - \frac{1}{108\cos^2\phi/3} + \frac{i\tan\phi/3}{54}\right) + O(k^6) \tag{3.2}$$

This result is to be compared with the fourth-order result on the superintegrable manifold, which may be obtained for arbitrary N. From a straightforward perturbation expansion we may reduce the correlation to a set of trigonometric sums

$$
\begin{aligned}
&\langle (Z_0 Z_1^\dagger)^n \rangle \\
&= 1 - \frac{n}{N}\left(1 - \frac{n}{N}\right) k^2 \\
&\quad + \left\{ \frac{1}{16N^4} \sum_{j=1}^{N-1} \sum_{k=1}^{N-1} \frac{2(1-\omega^{nj}) + 2(1-\omega^{-nk}) - (1-\omega^{n(j-k)})}{\sin^2(\pi j/N)\sin^2(\pi k/N)} \right. \\
&\quad - \frac{1}{8N^4} \sum_{p=1}^{N-1} \left(1 - \cos\frac{2\pi np}{N}\right) \\
&\quad \times \sum_{\substack{j=1 \\ j\neq p}}^{N-1} \sum_{\substack{k=1 \\ k\neq p}}^{N-1} \left[\sin\frac{\pi j}{N} \sin\frac{\pi(p-j)}{N} \sin\frac{\pi k}{N} \sin\frac{\pi(p-k)}{N} \right]^{-1} \\
&\quad - \frac{1}{4N^4} \sum_{p=1}^{N-1} \frac{1-\cos(2\pi np/N)}{\sin^4(\pi p/N)} + \frac{1}{4N^4} \sum_{p=1}^{N-1} \left(1 - \cos\frac{2\pi np}{N}\right) \\
&\quad \times \sum_{\substack{j=1 \\ j\neq p}}^{N-1} \sum_{\substack{k=1 \\ k\neq j}}^{N-1} \left[\sin\frac{\pi(p-j)}{N} \sin\frac{\pi k}{N} \sin\frac{\pi(j-k)}{N} \sin\frac{\pi p}{N} \right]^{-1} \\
&\quad \left. - \frac{3}{16N^4} \sum_{j=1}^{N-1} \sum_{k=1}^{j} \frac{(1-\omega^{n(j-k)}) + (1-\omega^{-nj}) + (1-\omega^{nk})}{\sin^2(\pi j/N)\sin^2(\pi k/N)} \right\} k^4 + O(k^6)
\end{aligned} \tag{3.3}
$$

The trigonometric sums have been evaluated in ref. 25 and thus the final result is

$$
\begin{aligned}
\langle (Z_0 Z_1^\dagger)^n \rangle - M_n^2 = \Bigg(& \frac{1}{2}\left[\frac{n}{N}\left(1-\frac{n}{N}\right)\right]^2 + \frac{1}{4N^2}\frac{n}{N}\left(1-\frac{n}{N}\right) \\
& - i\frac{3}{4N^4}\sum_{p=1}^{N-1} f(N,n,p)\tan\frac{\pi p}{2N}\Bigg) k^4 + O(k^6)
\end{aligned} \tag{3.4}
$$

where

$$
f(N,n,p) = \begin{cases} n^2(N-2p) & \text{if } 1 \leqslant n \leqslant p \leqslant N/2 \\ f(N,p,n) & \text{in general} \\ -f(N,N-n,p) & \text{in general} \end{cases} \tag{3.5}
$$

Thus if we set $N=3$ in (3.4), we obtain (3.2) with $\phi = \pi/2$.

4. SUM RULE FOR SUPERINTEGRABLE CORRELATION FUNCTIONS

The superintegrable nearest neighbor correlation functions $\langle (Z_0 Z_1^\dagger)^n \rangle$ obey a sum rule which provides a check of the perturbation expansion (3.4). Here we derive several forms for the sum rule.

We start by applying the Feynman–Hellmann formula[26]

$$\frac{\partial e^{(N)}}{\partial k} = \frac{1}{L} \langle 0| \frac{\partial H}{\partial k} |0\rangle \tag{4.1}$$

to (1.1) to obtain

$$\sum_{n=1}^{N-1} \langle 0| \alpha_n (Z_0 Z_1^\dagger)^n |0\rangle = -e^{(N)} + k \frac{\partial e^{(N)}}{\partial k} \tag{4.2}$$

where

$$\alpha_n = \frac{e^{i(2n-N)\phi/N}}{\sin(\pi n/N)} \tag{4.3}$$

and the ground-state energy per site of the N-state superintegrable chiral Potts chain is[8]

$$e^{(N)}(k) = -(1+k) \sum_{l=1}^{N-1} F\left(-\frac{1}{2}, \frac{l}{N}; 1; \frac{4k}{(1+k)^2}\right) \tag{4.4}$$

With the aid of ref. 27 [p. 102, (21)]

$$F'(a, b; c; z) = \frac{a}{z} [F(a+1, b; c; z) - F(a, b; c; z)] \tag{4.5}$$

(4.2) and (4.4) yield a sum rule

$$\begin{aligned}\sum_{n=1}^{N-1} \langle 0| \alpha_n (Z_0 Z_1^\dagger)^n |0\rangle = \sum_{n=1}^{N-1} \bigg[&\frac{1}{2}(1-k) F\left(\frac{1}{2}, \frac{n}{N}; 1; \frac{4k}{(1+k)^2}\right) \\ &+ \frac{1}{2}(1+k) F\left(-\frac{1}{2}, \frac{n}{N}; 1; \frac{4k}{(1+k)^2}\right)\bigg]\end{aligned} \tag{4.6}$$

We can reexpress the sum rule in terms of the functions $F(n/N, n/N; 1, k^2)$ and $F(-n/N, n/N; 1, k^2)$ by using the two identities

$$\begin{aligned}F\left(\frac{1}{2}, b; 1; \frac{4k}{(1+k)^2}\right) &= (1+k)^{2b} F(b, b; 1; k^2) \\ F\left(-\frac{1}{2}, b; 1; \frac{4k}{(1+k)^2}\right) &= (1+k)^{2b-2} \{4(1-b) F(b, -b; 1; k^2) \\ &\quad + [(4b-3)-k](1-k) F(b, b; 1; k^2)\}\end{aligned} \tag{4.7}$$

The first of these is just Eq. (36), p. 113, of ref. 27 with $a=b$. The second can be obtained from

$$F\left(-\frac{1}{2}, b; 1; \frac{4k}{(1+k)^2}\right) = (2b-1)\left(1-\frac{4k}{(1+k)^2}\right) F\left(\frac{1}{2}, b; 1; \frac{4k}{(1+k)^2}\right) + 2(1-b)\, F\left(\frac{1}{2}, b-1; 1; \frac{4k}{(1+k)^2}\right) \tag{4.8}$$

[Eq. (37), p. 103, of ref. 27 with $a=1/2$, $c=1$, and $z=4k/(1+k)^2$] by using the first identity and Eq. (33), p. 103, of ref. 27, with $a=b-1$, $c=1$, and $z=k^2$. The sum rule now becomes

$$\sum_{n=1}^{N-1} \langle 0|\, \alpha_n (Z_0 Z_1^\dagger)^n \,|0\rangle = \sum_{n=1}^{N-1} \left[\left(\frac{2n}{N}-1\right)(1-k^2)(1-k)^{2n/N-2}\, F\left(\frac{n}{N}, \frac{n}{N}; 1; k^2\right) + 2\left(1-\frac{n}{N}\right)(1+k)^{2n/N-1}\, F\left(\frac{n}{N}-1, \frac{n}{N}; 1; k^2\right)\right] \tag{4.9}$$

The two sides of the sum rule are even functions of k. This is made manifest by adding (4.9) with $n \to N-n$ to itself and using Eq. (2), p. 105, and Eq. (31), p. 103, of ref. 27 to obtain

$$\sum_{n=1}^{N-1} \frac{1}{\sin(\pi n/N)} \langle 0|\, e^{i\pi(2n-N)/2N} (Z_0 Z_1^\dagger)^n + e^{-i\pi(2n-N)/2N} (Z_0 Z_1^\dagger)^{N-n} \,|0\rangle$$
$$= \sum_{n=1}^{N-1} \left\{ 2\left(1-\frac{n}{N}\right)\left[(1+k)^{2n/N-1} + (1-k)^{2n/N-1}\right] F\left(\frac{n}{N}-1, \frac{n}{N}; 1; k^2\right) + \left(\frac{2n}{N}-1\right)(1-k^2)\left[(1+k)^{2n/N-2} + (1-k)^{2n/N-2}\right] \times F\left(\frac{n}{N}, \frac{n}{N}; 1; k^2\right)\right\} \tag{4.10}$$

Expanding the right-hand side of (4.10) to fourth order and using (3.4) in the left-hand side, we find that the k^4 term in the sum rule gives

$$\sum_{j=1}^{N-1} \sum_{k=1}^{N-1} f(N, j, k) \cot\frac{\pi j}{N} \cot\frac{\pi k}{N} = \frac{N(N^2-1)(N^2-4)}{90} \tag{4.11}$$

This identity is a special case of the more general identity, which is independently proven in Appendix A,

$$S(N)=\sum_{j=1}^{N-1}\sum_{k=1}^{N-1} f_Q(N,j,k)\cot\frac{\pi j}{N}\cot\frac{\pi k}{N}$$
$$=\frac{2(N-1)(N-2)}{3}\,Q(N,0)-4\sum_{j=1}^{\lfloor (N-1)/2\rfloor}(N-2j)\,Q\left(N,\frac{j}{N}\right)\tag{4.12}$$

where

$$f_Q(N,j,k)=\begin{cases}P(n,j,k) & \text{if}\quad 1\leqslant j\leqslant k\leqslant N/2\\ f_Q(N,k,j) & \text{in general}\\ -f_Q(N,N-j,k) & \text{in general}\end{cases}\tag{4.13}$$

and

$$P(N,j,k)=Q\left(N,\frac{k+j}{N}\right)+Q\left(N,\frac{k-j}{N}\right)-2Q\left(N,\frac{k}{N}\right)\tag{4.14}$$

for any function $Q(N,x)$ satisfying

$$Q(N,1-x)=-Q(N,x)\tag{4.15}$$

We note that (4.12) is a member of a broader class of identities, described in Appendix B.

We conclude this section by deriving an order $2N-2$ Fuchsian differential equation for the superintegrable ground-state energy (4.4) with three singularities located at 0, 1, and ∞. Equation (4.4) is expressed in terms of hypergeometric functions which are solutions to the second-order Fuchsian equation with singularities as above,

$$D_{(a,b;c;z)}F(a,b;c;z)=0\tag{4.16}$$

where

$$D_{(a,b;c;z)}=z(1-z)\frac{d^2}{d^2z}+[c-(a+b+1)z]\frac{d}{dz}-ab\tag{4.17}$$

One can readily verify that this differential operator satisfies

$$D_{(a+1,b_1;c;z)}D_{(a,b_2;c;z)}=D_{(a+1,b_2;c;z)}D_{(a,b_1;c;z)}\tag{4.18}$$

We also define the differential operator

$$D'_{(a,b;c;k)}=(1+k)\,D_{(a,b;c;4k/(1+k)^2)}(1+k)^{-1}\tag{4.19}$$

which has solution $(1+k)\,F(a, b; c; 4k/(1+k)^2)$ and which satisfies an equation precisely analogous to (4.18). Now let

$$D_N = D'_{((2N-5)/2,\,(N-1)/N;\,1;\,k)} D'_{((2N-7)/2,\,(N-2)/N;\,1;\,k)} \cdots D'_{(-1/2,\,1/N;\,1;\,k)} \quad (4.20)$$

By repeated application of (4.18) we see that

$$D_N = D'_{((2N-5)/2,\,b_{N-1}/N;\,1;\,k)} D'_{((2N-7)/2,\,b_{N-2}/N;\,1;\,k)} \cdots D'_{(-1/2,\,b_1/N;\,1;\,k)} \quad (4.21)$$

where $(b_1,..., b_{N-1})$ is any permutation of $(1,..., N-1)$. In particular, b_1 may take any value from 1 to $N-1$, from which it immediately follows that

$$D_N e^{(N)}(k) = 0 \quad (4.22)$$

5. DISCUSSION

No finite order of perturbation theory can ever stand as the final solution of any problem. However, such perturbation studies often suggest many questions for further study and we conclude by discussing some questions raised by the present study.

First it must be noted that the powers of the poles at any particular location increase with the power of k. This is an indication that the location of the true singularity depends on k and it would be most desirable to find a perturbation theory directly for the location of the singularities.

In terms of extending our results, we remark that while it may be cumbersome to extend the results for the nonintegrable case, it is clear that for the integrable and superintegrable cases a great deal more can be computed for the perturbation expansion of the correlation functions. It is to be hoped that further terms in these expansions will produce conjectures such as (3.1) for the full correlation functions.

Finally there is the question of whether or not the increased analyticity which results from the cancellation of poles can be taken as a sufficient condition for integrability, which can be considered as part of the problem of the general relation between analyticity and integrability.

The construction of models which satisfy the definition of integrability (1.5) in terms of commuting transfer matrices can be viewed as essentially a problem in algebra. Indeed one of the major developments in algebra in the last 15 years is quantum groups, which were invented specifically in order to find classes of solutions to (1.5). Analysis, on the other hand, seems to have only a marginal relation to (1.5) and, in the larger scheme of mathematics, algebra and analysis are almost universally treated as separate disciplines.

A corresponding separation of disciplines has been traditional in physics, where statistical mechanics has relied heavily on algebra, while field theory has relied on analysis. Thus, for example, while Onsager's solution of the Ising model[28] is totally algebraic, the renomalization theory of quantum electrodynamics is almost exclusively concerned with subtractions of infinities in integrals and seems to be totally concerned with analysis.

Thus 50 years ago integrability (algebra) and analyticity (analysis) were seen to be totally separate. However, in the 50 years that has followed these two totally separate computations, there has been a remarkable merging of field theory and statistical mechanics, which can be viewed as either the introduction of algebra into field theory or the introduction of analysis into statistical mechanics. Thus, as an example, at present we have the isomorphism in two dimensions of solvable statistical mechanical models (solved by algebra) with conformal quantum field theory (where analysis appears in the very statement of the subject).

If we take this merging of analyticity and integrability seriously, however, then we are forced to conclude that the recent four-dimensional studies on $N=2$ Yang–Mills theory,[3,4] which have total reliance on analyticity properties, should have an isomorphic counterpart whose solution relies on algebra. The discovery of such an algebraic counterpart (or the demonstration that it does not exist) to the work of refs. 3 and 4 will constitute an important advance in our understanding of the relation of integrability and analyticity.

APPENDIX A. PROOF OF TRIGONOMETRIC IDENTITY (4.12)

To prove (4.12) we begin by using the symmetries of $f(N, j, k)$ and the cotangent function to write

$$S(N)=4\sum_{j=1}^{\lfloor (N-1)/2\rfloor}\sum_{k=j+1}^{N-j-1} P(N,j,k)\cot\frac{\pi j}{N}\cot\frac{\pi k}{N} + 4\sum_{j=1}^{\lfloor (N-1)/2\rfloor} P(N,j,j)\cot^2\frac{\pi j}{N} \tag{A.1}$$

We recall that $P(N, j, k)$ was defined by (4.14) in terms of a function $Q(N, x)$ satisfying the symmetry property (4.15) and then write $Q(N, x)$ so that the symmetry property automatically holds,

$$Q(N,x)=r(x)-r(1-x) \tag{A.2}$$

where $r(x)$ is an arbitrary function whose N dependence has been suppressed for brevity. Using this definition, we expand (A.1) as

$$\begin{aligned}
S(N) = {} & 8 \sum_{j=1}^{\lfloor (N-1)/2 \rfloor} \sum_{k=j+1}^{N-j-1} \left[r\left(\frac{k+j}{N}\right) - r\left(\frac{k}{N}\right) \right] \cot\frac{\pi j}{N} \cot\frac{\pi k}{N} \\
& + 4 \sum_{j=1}^{\lfloor (N-1)/2 \rfloor} \left[r\left(\frac{2j}{N}\right) + r(0) + 2r\left(\frac{N-j}{N}\right) \right] \cot^2\frac{\pi j}{N} \\
& - 8 \sum_{j=1}^{\lfloor (N-1)/2 \rfloor} \sum_{k=j+1}^{N-j-1} \left[r\left(\frac{N-k-j}{N}\right) - r\left(\frac{N-k}{N}\right) \right] \cot\frac{\pi j}{N} \cot\frac{\pi k}{N} \\
& - 4 \sum_{j=1}^{\lfloor (N-1)/2 \rfloor} \left[r\left(\frac{N-2j}{N}\right) + r(1) + 2r\left(\frac{j}{N}\right) \right] \cot^2\frac{\pi j}{N}
\end{aligned} \tag{A.3}$$

which may be rewritten as

$$S(N) = \sum_{a=0}^{N} C(N, a)\, r\left(\frac{a}{N}\right) = \sum_{a=0}^{\lfloor (N-1)/2 \rfloor} C(N, a)\, Q\left(N, \frac{a}{N}\right) \tag{A.4}$$

where

$$C(N, a) = \begin{cases} 4 \displaystyle\sum_{j=1}^{\lfloor (N-1)/2 \rfloor} \cot^2\frac{\pi j}{N} & \text{for} \quad a = 0 \\ 4C_1(N, a) + 4C_2(N, a) & \text{for} \quad 1 \leqslant a \leqslant \lfloor (N-1)/2 \rfloor \\ -C(N, N-a) & \text{for all } a \end{cases} \tag{A.5}$$

with

$$\begin{aligned}
C_1(N, a) = {} & 2 \sum_{j=1}^{\lfloor (a-1)/2 \rfloor} \cot\frac{\pi j}{N} \cot\frac{\pi (a-j)}{N} \\
& - 2 \sum_{j=1}^{a-1} \cot\frac{\pi j}{N} \cot\frac{\pi a}{N} + \varepsilon_a \cot^2\frac{\pi a}{2N}
\end{aligned} \tag{A.6}$$

and

$$\begin{aligned}
C_2(N, a) = {} & -2 \sum_{j=1}^{\lfloor (N-a-1)/2 \rfloor} \cot\frac{\pi j}{N} \cot\frac{\pi (N-a-j)}{N} \\
& + 2 \sum_{j=1}^{a-1} \cot\frac{\pi j}{N} \cot\frac{\pi (N-a)}{N} \\
& - \varepsilon_{N-a} \cot^2\frac{\pi a}{2N} - 2\cot^2\frac{\pi a}{N}
\end{aligned} \tag{A.7}$$

Here

$$\varepsilon_a = \begin{cases} 1 & \text{if } a \text{ is even} \\ 0 & \text{if } a \text{ is odd} \end{cases} \tag{A.8}$$

The sum $C(N, 0)$ can be rewritten as

$$C(N, 0) = 2 \sum_{j=1}^{N-1} \cot^2 \frac{\pi j}{N} \tag{A.9}$$

which is recognized as the Dedekind sum[29] and hence

$$C(N, 0) = \frac{2(N-1)(N-2)}{3} \tag{A.10}$$

To evaluate $C_1(N, a)$ we first write

$$\begin{aligned} C_1(N, a) = & \sum_{j=1}^{\lfloor (a-1)/2 \rfloor} \cot \frac{\pi j}{N} \cot \frac{\pi (a-j)}{N} + \sum_{j=\lfloor (a+2)/2 \rfloor}^{a-1} \cot \frac{\pi (a-j)}{N} \cot \frac{\pi j}{N} \\ & + \varepsilon_a \cot^2 \frac{\pi a}{2N} - 2 \sum_{j=1}^{a-1} \cot \frac{\pi j}{N} \cot \frac{\pi a}{N} \\ = & \sum_{j=1}^{a-1} \cot \frac{\pi j}{N} \cot \frac{\pi (a-j)}{N} - 2 \sum_{j=1}^{a-1} \cot \frac{\pi j}{N} \cot \frac{\pi a}{N} \\ = & -\sum_{j=1}^{a-1} \cot \frac{\pi j}{N} \cot \frac{\pi (j+N-a)}{N} \\ & - 2 \cot \frac{\pi (N-a)}{N} \sum_{j=1}^{N-a} \cot \frac{\pi j}{N} \end{aligned} \tag{A.11}$$

Then let $\omega = \exp(2\pi i/N)$ and write the first summand as

$$\begin{aligned} & -\cot \frac{\pi j}{N} \cot \frac{\pi (j+N-a)}{N} \\ & \quad = \frac{\omega^j + 1}{\omega^j - 1} \cdot \frac{\omega^{j+N-a} + 1}{\omega^{j+N-a} - 1} \\ & \quad = \left[1 + \frac{2}{\omega^j - 1} \right] \left[1 + \frac{2}{\omega^{j+N-a} - 1} \right] \\ & \quad = \left[1 + \frac{2}{\omega^j - 1} + \frac{2}{\omega^{j+N-a} - 1} + 4 \frac{1}{\omega^j - 1} \cdot \frac{1}{\omega^{j+N-a} - 1} \right] \end{aligned}$$

$$
\begin{aligned}
&=\left[1+\frac{2}{\omega^{j}-1}+\frac{2}{\omega^{j+N-a}-1}+4\left(\frac{1}{\omega^{j}-1}-\frac{\omega^{N-a}}{\omega^{j+N-a}-1}\right)\frac{1}{\omega^{N-a}-1}\right]\\
&=\left[1+\frac{2}{(\omega^{j}-1)(\omega^{N-a}-1)}(\omega^{N-a}-1+2)\right.\\
&\qquad\left.+\frac{2}{(\omega^{j+N-a}-1)(\omega^{N-a}-1)}(\omega^{N-a}-1-2\omega^{N-a})\right]\\
&=\left[1+2\frac{\omega^{N-a}+1}{\omega^{N-a}-1}\left(\frac{1}{\omega^{j}-1}-\frac{1}{\omega^{j+N-a}-1}\right)\right]
\end{aligned}
\tag{A.12}
$$

The sum partially telescopes,

$$
\begin{aligned}
&-\sum_{j=1}^{a-1}\cot\frac{\pi j}{N}\cot\frac{\pi(j+N-a)}{N}\\
&\quad=(a-1)+2\frac{\omega^{N-a}+1}{\omega^{N-a}-1}\left[\frac{1}{\omega^{1}-1}+\frac{1}{\omega^{2}-1}+\cdots+\frac{1}{\omega^{N-a}-1}\right.\\
&\qquad\left.-\frac{1}{\omega^{N-1}-1}-\frac{1}{\omega^{N-2}-1}-\cdots-\frac{1}{\omega^{a}-1}\right]\\
&\quad=(a-1)+2\frac{\omega^{N-a}+1}{\omega^{N-a}-1}\left[\frac{\omega^{1}+1}{\omega^{1}-1}+\frac{\omega^{2}+1}{\omega^{2}-1}+\cdots+\frac{\omega^{N-a}+1}{\omega^{N-a}-1}\right]\\
&\quad=(a-1)+2\cot\frac{\pi(N-a)}{N}\sum_{j=1}^{N-a}\cot\frac{\pi j}{N}
\end{aligned}
\tag{A.13}
$$

It follows that

$$
C_{1}(N,a)=a-1 \tag{A.14}
$$

A parallel computation shows that

$$
\begin{aligned}
C_{2}(N,a)&=\sum_{j=1}^{N-a-1}\cot\frac{\pi j}{N}\cot\frac{\pi(j+a)}{N}-2\cot\frac{\pi a}{N}\sum_{j=1}^{a}\cot\frac{\pi j}{N}\\
&=-C_{1}(N,N-a)=-(N-a-1)
\end{aligned}
\tag{A.15}
$$

and therefore for $a\neq 0$

$$
C(N,a)=-4(N-2a) \tag{A.16}
$$

APPENDIX B. ADDITIONAL IDENTITIES RELATED TO (4.12)

The cotangents on the left-hand side of (4.11) may be expanded in powers of $\omega=\exp(2\pi i/N)$, but the roots of unity all cancel in the sum. We expect that similar cancellations will occur when the sum rule for the superintegrable correlation functions is applied to orders higher than four. Thus we have investigated the generalization of (4.11)

$$S(N) = \sum_{j=1}^{N-1} \sum_{k=1}^{N-1} h_P(N, j, k) \cot \frac{\pi j}{N} \cot \frac{\pi k}{N} = \text{polynomial in } N \quad \text{(B.1)}$$

where

$$h_P(N, j, k) = \begin{cases} P(N, j, k) & \text{if } 1 \leqslant j \leqslant k \leqslant N/2 \\ h_P(N, k, j) & \text{in general} \\ -h_P(N, N-j, k) & \text{in general} \end{cases} \quad \text{(B.2)}$$

and where $P(N, j, k)$ is a polynomial in N, j, and k of total degree d. In the case where N is prime, the only way for the roots of unity to vanish is for the coefficients of $\omega, \omega^2, ..., \omega^{N-1}$ all to be equal. This implies a system of equations for $P(N, j, k)$ which is overdetermined when N is sufficiently large. Remarkably, there are many solutions to this system and these solutions hold for all values of N tested. We conjecture that they are true for all N.

Up to the largest degree tested, $d = 14$, the number of solutions to the overdetermined system is equal to the number of solutions to

$$d = 3f + 2n \quad \text{(B.3)}$$

with f a positive integer and n a nonnegative integer. Thus we make the further conjecture that there are infinitely many solutions which can be organized into families, $f = 1, 2, 3, ...$, whose members, labeled by the index n, have degree given by (B.3). Although there is immense freedom in combining solutions, it has proven to be possible to write the solutions in such a form that general formulas, given below, can be guessed in the cases $f = 1, 2$. The solutions $P_{f,n}(N, j, k)$ that we have found for $f = 3, 4$ are shown in Table VIII and the corresponding sums $S_{f,n}(N, j, k)$, are shown in Table IX.

The first family, $f = 1$, comprises polynomials of the form (4.14). If in Eq. (4.12) we make the choice $Q(N, x) = N^d q_d(x)$ with $q_d(x)$ a polynomial of degree d, then $S(N)$ is a polynomial in N of degree $d + 2$. The symmetry property on $Q(N, x)$, (4.15), requires d to be odd and $d = 1$ implies $P(N, j, k) = 0$. Thus these solutions have the degree stated in (B.3).

Table VII. The First Five Sums $S_{2,n}(N)$ from (B.1) with $P_{2,n}(N, j, k)$ Given by (B.4)

$$S_{2,0}(N) = -N^2(N^2-1)(N^2-4)(53N^2+13)/(16\cdot 27\cdot 5\cdot 7)$$
$$S_{2,1}(N) = N^2(N^2-1)(N^2-4)(821N^4+280N^2+69)/(8\cdot 81\cdot 25\cdot 7)$$
$$S_{2,2}(N) = -N^2(N^2-1)(N^2-4)(30874N^6+11150N^4+3501N^2+845)/(32\cdot 81\cdot 5\cdot 7\cdot 11)$$
$$S_{2,3}(N) = N^2(N^2-1)(N^2-4)(103801414N^8+37967420N^6+12484059N^4+3718690N^2 +882407)/(8\cdot 27\cdot 125\cdot 7\cdot 11\cdot 13\cdot 17)$$
$$S_{2,4}(N) = -N^2(N^2-1)(N^2-4)(1012869789N^{10}+371602425N^8+123446295N^6 +38256875N^4+11003386N^2+2579010)/(32\cdot 81\cdot 5\cdot 7\cdot 11\cdot 13\cdot 31)$$

Table VIII. The First Three Members of the Third Family and the First Two Members of the Fourth Family of Solutions to (B.1) Expressed in Terms of the Bernoulli Polynomials and Using the Notation $J=j/N$, $K=k/N$

$$P_{3,0}(N,j,k) = N^9(J^4(K-2K^3+K^5)-2J^5(K-K^3)+J^6(K+\tfrac{1}{3}K^3))$$
$$P_{3,1}(N,j,k) = N^{11}(\tfrac{1}{4}J^4(K-6K^3+9K^5-4K^7)+\tfrac{3}{2}J^5(K^3-K^5)-\tfrac{1}{4}J^6(6K-9K^3+7K^5) +2J^7(K-K^3)-\tfrac{1}{4}J^8(3K+K^3))$$
$$P_{3,2}(N,j,k) = N^{13}(J^4(\tfrac{5}{9}B_9(K)+B_7(K))+J^5(\tfrac{1}{12}B_7(K)-\tfrac{1}{36}B_3(K))+J^6(2B_7(K)+\tfrac{41}{24}B_5(K) -\tfrac{1}{8}B_3(K))+\tfrac{3}{4}J^7B_5(K)+J^8(\tfrac{4}{3}B_5(K)+\tfrac{41}{72}B_3(K)-\tfrac{1}{16}B_1(K))+\tfrac{5}{9}J^9B_3(K) +J^{10}(\tfrac{1}{9}B_3(K)+\tfrac{1}{9}B_1(K)))$$
$$P_{4,0}(N,j,k) = N^{12}(J^5(K-3K^3+3K^5-K^7)-3J^6(K-2K^3+K^5)+J^7(3K-2K^3-K^5) -J^8(K+K^3))$$
$$P_{4,1}(N,j,k) = N^{14}(J^5(\tfrac{5}{9}B_9(K)+B_7(K))+J^6(\tfrac{2}{3}B_7(K)+\tfrac{1}{2}B_5(K)-\tfrac{1}{18}B_3(K))+J^7(\tfrac{14}{9}B_7(K) +\tfrac{17}{9}B_5(K))+J^8(\tfrac{4}{3}B_5(K)+\tfrac{1}{2}B_3(K)-\tfrac{1}{36}B_1(K))+J^9(\tfrac{5}{9}B_5(K)+\tfrac{20}{27}B_3(K)) +J^{10}(\tfrac{2}{9}B_3(K)+\tfrac{1}{9}B_1(K)))$$

Table IX. The Sums (B.1) Corresponding to the Polynomials in Table VIII

$$S_{3,0}(N) = N(N^2-1)(N^2-4)(N^6+5N^4+54N^2-80)\cdot 32/(27\cdot 25\cdot 7\cdot 11)$$
$$S_{3,1}(N) = N(N^2-1)(N^2-4)(9N^8+45N^6+46N^4+12920N^2-31680)\cdot 8/(27\cdot 25\cdot 7\cdot 11\cdot 13)$$
$$S_{3,2}(N) = -N(N^2-1)(N^2-4)(198N^{10}+990N^8+7980N^6+58105N^4+15362N^2 -786240)/(64\cdot 243\cdot 25\cdot 49\cdot 11\cdot 13)$$
$$S_{4,0}(N) = N^2(N^2-1)(N^2-4)(2N^8+10N^6+42N^4+2315N^2 -9614)\cdot 256/(27\cdot 25\cdot 49\cdot 11\cdot 13)$$
$$S_{4,1}(N) = -N^2(N^2-1)(N^2-4)(9N^{10}+45N^8+189N^6+7915N^4+11792N^2 -240240)/(32\cdot 243\cdot 25\cdot 49\cdot 11\cdot 13)$$

The second family of solutions is given by

$$P_{2,n}(N,j,k) = N^{2n+6} \sum_{h=0}^{n} \left[\left(\frac{j}{N}\right)^{2n+3-2h} b_{nh} B_{2h+3}\left(\frac{k}{N}\right) + \left(\frac{j}{N}\right)^{2n+2-2h} e_{nh} E_{2h+3}\left(\frac{k}{N}\right) \right] \tag{B.4}$$

where, letting $g(n) = 4^{n+1} - 1$,

$$\begin{aligned} b_{nh} &= 2\binom{2n+3}{2h} \frac{1}{(2h+1)(2h+2)(2h+3)} \frac{g(h)\, g(n-h)}{g(n)} \\ e_{nh} &= 2\binom{2n+3}{2h+1} \frac{1}{(2h+2)(2h+3)} \frac{g(h)\, 4^{n-h}}{g(n)} \end{aligned} \tag{B.5}$$

and where $B_n(x)$ and $E_n(x)$ are the Bernoulli and Euler polynomials. The sum in (B.4) may be performed using the identities[30]

$$\begin{aligned} B_n(x+h) &= \sum_{k=0}^{n} \binom{n}{k} B_k(x)\, h^{n-k} \\ E_n(x+h) &= \sum_{k=0}^{n} \binom{n}{k} E_k(x)\, h^{n-k} \end{aligned} \tag{B.6}$$

to obtain

$$\begin{aligned} P_{2n} = \frac{N^{2n+6}}{(2n+6)(2n+5)(2n+4)(4^{n+1}-1)} \Bigg[& (2\cdot 4^{n+1}+1)\, B_{2n+6}\left(\frac{k+j}{N}\right) \\ & - (6\cdot 4^{n+1}+1)\, B_{2n+6}\left(\frac{k-j}{N}\right) + (4^{n+2}-1)\, B_{2n+6}\left(\frac{k}{N}\right) \\ & + B_{2n+6}\left(\frac{k-2j}{N}\right) \\ & - 2\cdot 4^{n+2}\left(B_{2n+6}\left(\frac{k+j/2}{N}\right) - B_{2n+6}\left(\frac{k-j/2}{N}\right)\right) \\ & + 2\cdot 4^{2n+4}\left(B_{2n+6}\left(\frac{k+j}{2N}\right) + B_{2n+6}\left(\frac{k-j}{2N}\right)\right) \\ & - 4^{n+3}(4^{n+2}-1)\, B_{2n+6}\left(\frac{k}{2N}\right) \\ & - 2\cdot 4^{n+2}\left(B_{2n+6}\left(\frac{k+2j}{2N}\right) + B_{2n+6}\left(\frac{k-2j}{2N}\right)\right)\Bigg] \end{aligned} \tag{B.7}$$

This expression differs from (4.14) in that the Bernoulli polynomials cannot be replaced by arbitrary polynomials satisfying $q(1-x) = -q(x)$. A general form has not been found for the corresponding sums $S_{2,n}(N)$. The first five are shown in Table VII.

ACKNOWLEDGMENTS

The authors wish to thank Prof. Gert Almkvist, Prof. G. von Gehlen, Prof. A. Guttmann, and Dr. A. Honecker for insightful discussions. This work was partially supported by NSF grant DMR 9404747.

REFERENCES

1. R. J. Baxter, in *Fundamental Studies in Statistical Mechanics V*, E. G. D. Cohen, ed. (North-Holland, Amsterdam, 1980), p. 109.
2. R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, 1982).
3. N. Seiberg, *Phys. Lett. B* **318**:469 (1993); *Phys. Rev. D* **49**:6857 (1994).
4. N. Seiberg and E. Witten, *Nucl. Phys. B* **426**:12 (1994); **431**:482 (1994).
5. H. Au-Yang, B. M. McCoy, J. H. H. Perk, S. Tang, and M.-L. Yan, *Phys. Lett. A* **123**:219 (1987).
6. R. J. Baxter, J. H. H. Perk, and H. Au-Yang, *Phys. Lett. A* **128**:138 (1988).
7. H. Au-Yang and J. H. H. Perk, *Adv. Stud. Pure Math.* **19**:57 (1989).
8. G. Albertini, B. M. McCoy, and J. H. H. Perk, *Adv. Stud. Pure Math.* **19**:1 (1989).
9. R. J. Baxter, *Phys. Lett. A* **146**:110 (1990); Free energy of the chiral Potts model in the scaling region, Canberra preprint (1995).
10. R. J. Baxter, V. V. Bazhanov, and J. H. H. Perk, *Int. J. Mod. Phys. B* **4**:803 (1990).
11. G. Albertini, B. M. McCoy, J. H. H. Perk, and S. Tang, *Nucl. Phys. B* **314**:159 (1989).
12. G. Albertini, B. M. McCoy, and J. H. H. Perk, *Phys. Lett. A* **135**:159 (1989).
13. B. M. McCoy and W. P. Orrick, *Nucl. Phys. B (Proc. Suppl.) A* **45**:50–61 (1996).
14. J. J. Sakurai, *Modern Quantum Mechanics* (Addison-Wesley, Reading, Massachusetts, 1985).
15. N. S. Han and A. Honecker, *J. Phys. A* **27**:9 (1994).
16. A. Honecker, Quantum spin models and extended conformal algebra [Hepth 9503104].
17. A. Honecker and G. von Gehlen, *Nucl. Phys. B* **435**:505 (1985).
18. A. Honecker, A perturbative approach to the chiral Potts model [Hepth 9409122].
19. I. G. Enting and A. J. Guttmann, The Ising model, self avoiding walks and solvability, Lattice '95 Proceedings, *Nucl. Phys. B Suppl.*, submitted.
20. T. T. Wu, B. M. McCoy, C. A. Tracy, and E. Barouch, *Phys. Rev. B* **13**:316 (1976).
21. C. A. Tracy, in *Nonlinear Equations in Physics and Mathematics*, A. O. Barut, ed. (Reidel, Dortrecht, Holland, 1978), p. 221.
22. R. J. Baxter, *Phil. Trans. R. Soc. Lond.* **289**:315 (1978).
23. R. J. Baxter, *J. Stat. Phys.* **63**:433 (1991); **70**:535 (1993).
24. S. Howes, L. P. Kadanoff, and M. den Nijs, *Nucl. Phys. B* **215**[FS7]:169 (1983).
25. A. Gervois and M. L. Mehta, *J. Math. Phys.* **36**:5098 (1995).
26. S. Gasiorowicz, *Quantum Physics* (Wiley, New York, 1974).
27. A. Erdelyi, *Higher Transcendental Functions*, Vol. 1 (McGraw-Hill, New York, 1953).
28. L. Onsager, *Phys. Rev.* **65**:117 (1944).
29. H. Rademacher and E. Grosswald, *Dedekind Sums* (Mathematical Association of America, 1972).
30. M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (U. S. Government Printing Office, 1972).